Introduction to Cosmology

A Brief Summary

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0.1 Introduction

This guide has been written as a complement for anyone who wishes to study or review the wonderful field of cosmology. It begins by discussing some of the observational features of our universe and how this relates to the field, before moving into the more theoretical realms. I acknowledge that this guide is a work in progress, and I don't expect a near-complete guide to be here for another few months at the least. Even then, the field is so beautifully broad and complex that no single guide could ever do it justice. The main inspiration for the content of this guide is my former Cosmology professor Neta Bahcall, who taught a wonderful introductory course. Many thanks also goes to Paul Steinhardt, who taught a delightful advanced course.

This guide is not yet complete. I hope to have it completed within the next month or two. .

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0.2 Appendix

A set of useful equations
[To be created soon]

Chapter 1

Observational Cosmology

1.1 Galaxies

The field of cosmology is so wondrously wide if can be difficult to know where to begin to start! We have to start somewhere, so just like my first cosmology course, let us explore the topic of galaxies. First of all, what do we mean when we talk about galaxies? Let's look at some of their properties to get a rough sense of them:

- Galaxies are a gravitationally bound group of stars, dust, matter etc.
- The galaxy we are in is the Milky Way. This is a typical galaxy, consisting of 1×10^{11} stars.
- The stars and matter are held in equilibrium.
- They contain huge amounts of dark matter. We shall discuss this point in far more detail soon.

The stars we see in the night sky are all contained in our galaxy, and were once believed to make up the entire Universe. In 1920, there was a major debate (the Shapley-Curtis debate) as to whether there exist galaxies other than our own. Needless to say, there are! Of course, we need strong telescopes in order to see these galaxies. Planets like our own move at about 200kms^{-1} around the center of the Milky Way and the period of rotation is 2×10^8 years.

Continuing with this topic, there are numerous different types of galaxies. More specifically, there are three general types of galaxies in the Morphological Classification. These are Elliptical, Spiral, and Irregular These can be represented by the Hubble tuning fork as shown above. The classifications are more rigorous than you may infer from the somewhat arbitrary naming system. For example, For Elliptical Galaxies, $E_0, E_1, ..., E_7$; where for $E_n, n = 10(1 - b/a)$. Here, a is the semi-major axis and b is the semi-minor axis. As well as galaxies, there are **globular clusters**, which are groups of $\approx$ 100,000 stars, typically contained within galaxies and held together by gravity. They are often held in a stable orbit around a galaxy.

1.1.1 Elliptical Galaxies

Let us continue by considering some of the features of elliptical galaxies. Unlike spiral galaxies, elliptical galaxies have no spiral arms. They tend to be far older than spiral galaxies, in fact, spiral galaxies tend to evolve into elliptical galaxies. From this, you can easily remember and explain many of the other properties of elliptical galaxies. First of all, there is very little gas contained within them. This is because the gas has been mostly used up in star formation earlier in the galaxy's lifetime. Because there is little gas, there is also little star formation and very few young stars. Because all of the stars are old, elliptical galaxies tend to

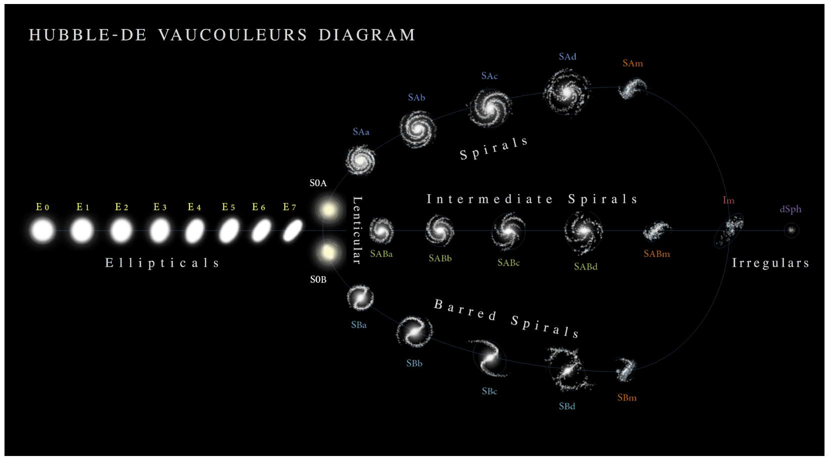

Figure 1.1: The Hubble Tuning Fork, sourced from Wikipedia.

be red. There are some other useful facts worth noting: elliptical galaxies have randomised velocities with no distinct motion, and they have a large bulge-to-disc ratio. They are sometimes referred to as 'early-type galaxies' from the mistaken belief that elliptical galaxies formed earlier than spiral galaxies.

1.1.2 Spiral Galaxies

The other major type of galaxy that you should be well acquainted with are spiral galaxies. In general, many of the properties are opposite that of elliptical galaxies. To begin with, spirals have more structure than ellipticals, which is clear just looking at them. They are younger, have a high quantity of gas and a high rate of star formation, and are commonly blue as a result. In fact, There is more gas and dust the further along the Hubble tuning fork you go, and so increased star formation rate the further along the Hubble tuning fork you go. With regards to the structure of spiral galaxies, note that the Bulge-to-Disk ratio decreases the further along the Hubble tuning fork you go, The arms loosen out the further along the Hubble tuning fork you you go, and the bulges of spiral galaxies do not rotate. They are sometimes referred to as 'late-type galaxies'.

1.1.3 Distribution of Galaxies

For reference, there are far more spiral galaxies than elliptical galaxies. Spiral galaxies make up $\approx 70\%$. while S_0 galaxies account for 20%, and elliptical galaxies only 10%.

1.2 Density-Morphology Relation

The density-morphology relation is an observed trend in the density of galaxies. More specifically, in denser areas of the universe, there are more elliptical galaxies that cluster together and fewer spiral galaxies. This suggests that at high densities star formation is suppressed due to various tidal forces/gravitational interactions. The graph is shown below:

1.2.1 Evolution

There are two major mechanisms by which galaxies evolve over time:

1. Static evolution - Blue galaxy stars exhaust all gas and eventually turn red.

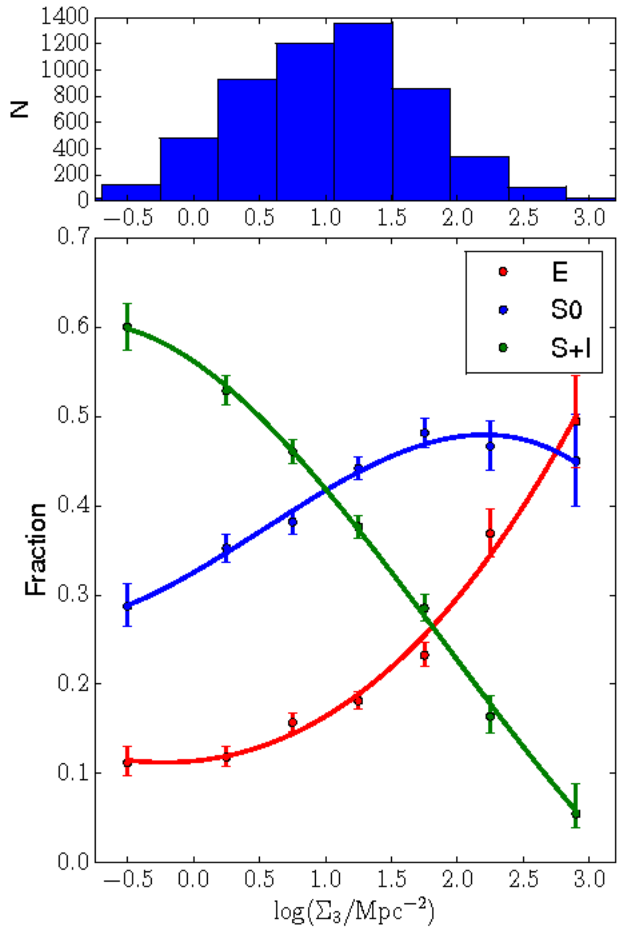

Figure 1.2: The Density Morphology relation. Sourced from http://inspirehep.net/record/1371676

2. Merge evolution - Gravity causes small galaxies to merge, or small galaxies to fall into larger galaxies. In this case, star formation dies out quickly and the galaxies become red, but there is an initial increase in luminosity.

1.3 Surface Brightness

Let us discuss surface brightness, which can be an important part of galaxy modelling.

$$I(r) = \frac{f}{\theta^2} \propto \frac{L/d^2}{R^2/d^2} \propto \frac{L}{R^2} \tag{1.1}$$

We can see from the above equation that surface brightness does not depend on distance from the observer. For the **Bulge** of a galaxy we have the following surface brightness:

$$I_{bulge}(r) = \frac{I_0}{(1 + r/r_c)^2} \tag{1.2}$$

where r_c is the "core radius" which is typically 0.3 - 1 Kpc.

For the disk of a galaxy we have another surface brightness equation,

$$I_{disk}(r) = I_s exp(-r/r_s) \tag{1.3}$$

where r_s is the "disk scale" which is typically 5 - 10 Kpc.

1.4 Colours

Let us consider the colour index of an object. A colour index is the difference in magnitude in an object when viewed from two different filters, e.g: B-V or R-I. The larger the magnitude, the fainter the object. The larger the luminosity, the brighter the object. A larger B-V colour index implies that the object is redder.

- B-V = 1 Elliptical galaxy (red)
- B-V < 1 Spiral galaxy (blue)

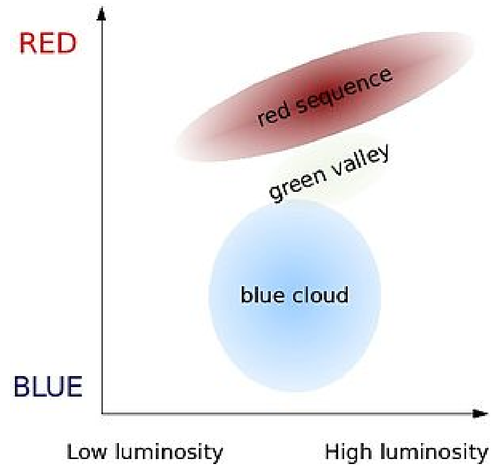

Figure 1.3: The Galaxy Colour-Magnitude Diagram. Galaxies typically "evolve" from blue to red cloud galaxies. The red cloud contains mostly elliptical galaxies and the blue cloud contains mostly spiral galaxies.

1.5 Doppler Shift

You are probably already familiar with the idea of redshift, where the wavelength of light from an object increases. There are three major types of redshift: the Doppler effect, cosmological redshift, and gravitational redshift. To begin, let us discuss the Doppler effect kind, which occurs when two objects are moving away each other. We can only measure the component of the movement which is radially away from us, so we shall ignore the transverse Doppler effect in the following discussion.

Redshift can be defined as follows:

$$z = \frac{\lambda - \lambda_0}{\lambda_0}, \tag{1.4}$$

$$1 + z = \frac{\lambda}{\lambda_0}. \tag{1.5}$$

Relativistic doppler shift takes the form:

$$\frac{\lambda}{\lambda_0} = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}, \tag{1.6}$$

and for v $\ll$ c you get equation (4) or,

$$z \approx v/c. \tag{1.7}$$

For elliptical galaxies, this is equivalent to saying

$$\frac{\sigma_r}{c} = \Delta z \tag{1.8}$$

where σ_r is radial dispersion. If one wants to apply this to spiral galaxies as well, then one needs to consider the movement relative to the observed wavelength, λ:

$$\frac{\Delta\lambda}{\lambda} c = \Delta v_{rotational} \tag{1.9}$$

Using equation (7) we can find the velocity with which the galaxy is moving away from us. Note that every absorption line will shift by the same amount.

As v $\rightarrow$ c then z $\rightarrow \infty$.

There is a second type of redshift that is highly important is cosmology, and is aptly named cosmological redshift. We shall discuss this more in the theoretical cosmology section, but effectively involves the space itself between objects expanding, causing redshift. Let us discuss the observational basis for this, which forms one of the strongest ideas for the origin of the universe.

In 1929, Hubble used measured velocities of the distant galaxies and plotted them against the distance, d_H, of each galaxy from Earth. He calculated the distances using Cepheid variables. The further away the galaxies are, the faster the galaxies are moving away.

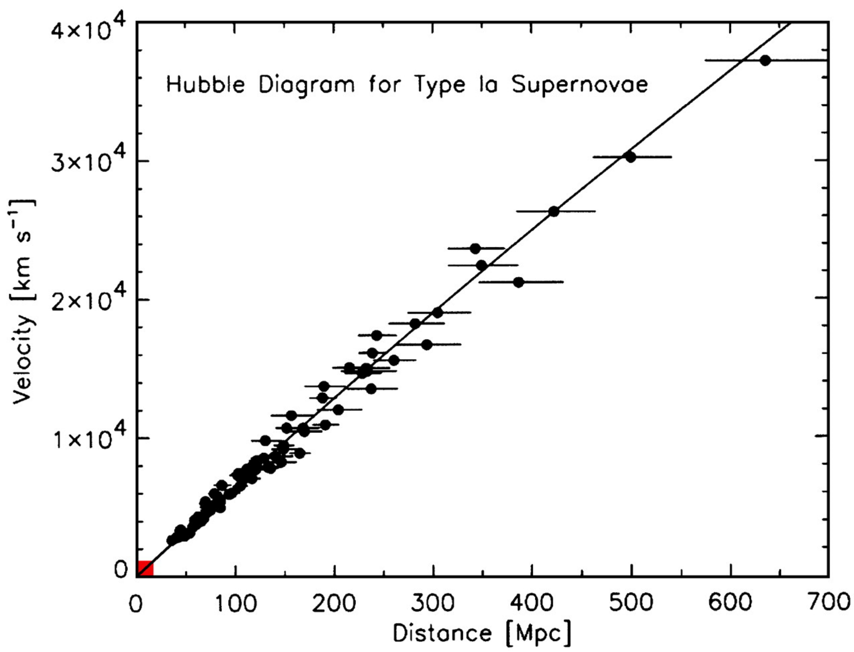

Figure 1.4: Hubble defined v = H_0d (Hubble's Law), where $H_0 \approx 70$ kms^{-1} Mpc $^{-1}$. Source: https://www.pnas.org/content/101/1/8

We can use the Hubble constant, H_0 to find the approximate age of the Universe, T_H:

$$T_H = \frac{1}{H_0}. \tag{1.10}$$

We can also use the Hubble constant and redshift to find the Hubble distance by using the relation,

$$d_H = \frac{cz}{H_0}. \tag{1.11}$$

$$v_{observed} = v_H + v_{peculiar} \tag{1.12}$$

The peculiar velocity of a galaxy is its velocity relative to the motion due to the isotropic expansion of the universe as described by the Hubble Flow.
For elliptical galaxies, we assume that,

$$v_x = v_y = v_z,$$

therefore,

$$v_{3D} = \sqrt{3}\,\sigma_r \tag{1.13}$$

For spiral galaxies we note that:

$$v_{rotation} = \frac{v_{observed}}{\sin\theta}, \tag{1.14}$$

where θ is the angle deviation from the galaxy being face on. So $\theta = 0$ means the galaxy is face on.

1.6 Rotation Curves

Galaxies are stable due to the balance of the gravitational force and the centrifugal force. Let us set these forces equal to one other and see what happens:

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \tag{1.15}$$

$$M(\leq R) = \frac{v^2 R}{G} \tag{1.16}$$

$$v \propto \frac{\sqrt{M(\leq R)}}{R} \tag{1.17}$$

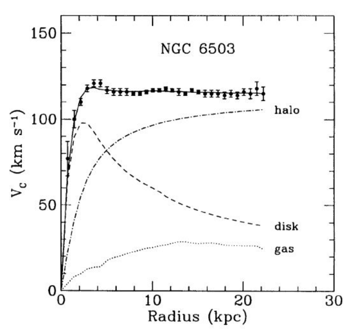

Figure 1.5: Rotation curve showing the existence of Dark Matter. The Dark Matter component extends to $\approx 10R_{luminous}$. Source: https://ned.ipac.caltech.edu/level5/Sept17/Freese/Freese2.html

Equation 1.17 is known as the virial equation. The extra mass in the galaxy is about 10 times the stellar mass and so we must have a certain amount of mass to compensate for the expected drop of $R^{1/2}$ because $v \propto$ M. Therefore,

$$M_{DM}(\leq R) \propto R \tag{1.18}$$

$$\rho_{DM} \propto \frac{M}{R^3} \propto \frac{1}{R^2}. \tag{1.19}$$

For almost all galaxies there exists a large dark matter halo which extends to a radius of $\approx$ 10 $R_{luminous}$. The nature of Dark Matter eludes astrophysicists today, and there are a whole number of ideas. We shall discuss this in more depth. Dark Matter is very likely non-baryonic. The mass density of the universe is $\approx$ 4% baryonic matter (an upper limit established by various experimental methods) and $\approx$ 20% non-baryonic dark matter. We shall discuss what the remaining 75% is soon.

1.7 The Tully-Fisher Relation

We need to be able to test the Hubble Law by measuring a true distance in another way. Hubble did this using standard candles as distance indicators. In 1977 Tully and Fisher measured the flux of **spiral galaxies** and showed that for a given waveband, there exists a relationship between velocity and luminosity. For a given waveband:

$$L_{band} \propto v_{rot}^4. \tag{1.20}$$

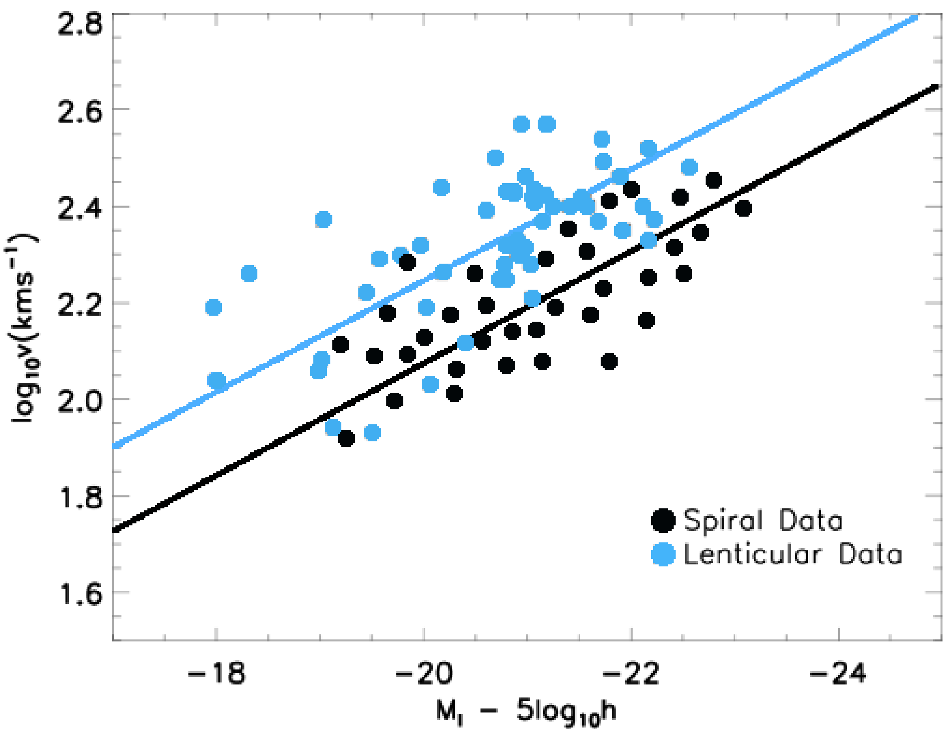

Figure 1.6: The Tully-Fisher relationship method. Source: Wikipedia

1.8 Faber-Jackson Relation

The Faber-Jackson relation is simply the analogy of the Tully-Fisher relation for elliptical galaxies.

$$L_{(band)} \propto v_{3D}^4 \tag{1.21}$$

Let us seek to understand where this comes from:
[add description]

1.9 Luminosity Function

The luminosity function of galaxies is the number of galaxies in a volume δV with luminosity L to L + δL. Recall that $L = 4\pi d^2 f$:

$$\delta N = \Phi(L)\delta V \delta L \tag{1.22}$$

$$\Phi(L) = \left(\frac{n_0}{L^*}\right)\left(\frac{L}{L^*}\right)^{\alpha} exp\left(\frac{L}{L^*}\right), \tag{1.23}$$

which has units of galaxy $L_{\odot}^{-1}\text{Mpc}^{-3}$. n_0 is the number density of L^* galaxies per unit volume. The above equation is called the **Schechter Luminosity Function.** The typical values of the constants in this luminosity function are as follows:

$L^* = 2 \times 10^{10} L_{\odot}$
$n_0 \approx 0.01$ gal L^* / Mpc^3.
$\alpha \approx$ -1 to -1.5

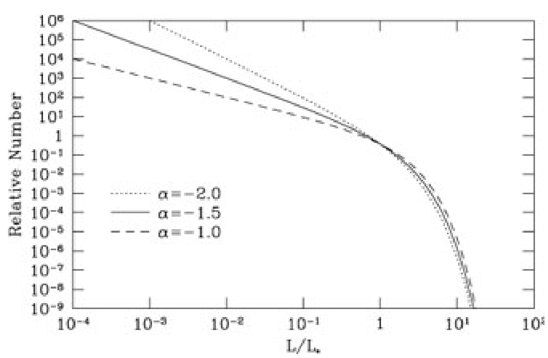

Figure 1.7: Schechter-Luminosity Graph. The Milky Way would be at an L / L* value ≈ 1. Source: https://ned.ipac.caltech.edu/level5/March01/Schneider/Schneider3.html

To find the total number of galaxies above a given luminosity,

$$N_{gal}(\geq L) = \int_L^{\infty} \Phi(L')dL' = n_0\Gamma(1+\alpha, L/L^*). \tag{1.24}$$

To find the total Luminosity density of the universe:

$$L_{total} = \int_0^\infty \Phi(L) L dL = n_0 L^*. \quad (1.25)$$

where we find that $L_{total} = 2 \times 10^8$ $L_\odot$ Mpc^{-3}.

This can be converted to a mass density if we know how much mass is associated with a unit of luminosity. We know from observation that for scales of $\approx$ 1 Mpc, M/L = 200 $M_\odot/L_\odot$, therefore,

$$L_{total}\frac{M}{L} = \rho_M = 4 \times 10^{10} M_\odot Mpc^{-3}, \quad (1.26)$$

where M/L is measured in a way such that we include dark matter (for example, using weak lensing). We can define a new quantity Ω_M,

$$\Omega_M \equiv \frac{\rho_M}{\rho_{crit}}, \quad (1.27)$$

where

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}. \quad (1.28)$$

This will be highly useful to us later on when exploring theoretical cosmology.

1.10 Clusters of Galaxies

A galaxy is by no means the smallest structure in the universe. In fact, there exist galaxy clusters. These are viralised groups of galaxies that are gravitationally, held in equilibrium. In fact, galaxy clusters are the largest virialised groups in the universe (superclusters, which are clusters of galaxy clusters, are defined by their not being viralised). The motion of galaxy clusters is somewhat similar to stars within an elliptical galaxy; there is no net rotation of galaxy clusters, just random motion of galaxies. Typically, in a galaxy cluster, there is one large galaxy at the center with a collection of small galaxies around. The large central galaxy is often the brightest, and is known as a Brightest Cluster Galaxy (BCG). The typical radius of a galaxy cluster is about 1 Mpc, with a typical mass of $10^{15} M_{sun}$.

1.10.1 Cluster Composition

It is important to know the general composition of clusters. Perhaps somewhat surprisingly, the stellar component of galaxies make up just 2% of the cluster mass. The Intra Cluster Medium (or ICM), which is just hot gas, accounts for about 13% of the mass of the cluster. The remaining 85% of the mass is dark matter. In the next section, we shall continue to explore the ICM.

1.11 Intra Cluster Medium

The Hot Gas within clusters emits X-rays (Brehmstrahlung radiation, caused by free-free electron scattering), and is approximately 10^7 - 10^8 degrees K. The high temperature relates to the kinetic energy of the gas, so we can calculate the velocities of the gas.

Electron density of the universe $n_e = 10^{-2}$cm^{-3}

$\text{Luminosity}_X \approx 10^{43}$ to $10^{45} \approx 10^{10}$ to 10^{12} $L_\odot$.

$$L_X = n_e^2 R_X^3 T^{0.5} \left(e^{-E1/kT} - e^{-E2/kT} \right) \tag{1.29}$$

where R_X is the radius of the cluster.

1.12 Dark Energy

One of the most interesting observations to be made in modern times (in 1997) was that the expansion of the universe is actually accelerating. This suggests that there must be a repulsive force in the universe that cancels out the gravitationally attractive forces of baryonic matter and dark matter. We call this dark energy. The make-up of the universe is as follows:

- 5% baryons
- 20% dark matter
- 75% dark energy

The density parameter, Ω , is defined as the ratio of the actual (or observed) density ρ to the critical density ρ_c of the Friedmann universe. The relation between the actual density and the critical density determines the overall geometry of the universe; when they are equal, the geometry of the universe is flat (Euclidean).

$$\frac{\Omega_{baryons}}{\Omega_M} = f_{baryons} = 0.15 \tag{1.30}$$

$$f^* = \frac{\Omega^*}{\Omega_M} = 0.02 \tag{1.31}$$

For small distances we can use the Hubble distance, but for larger distances we need a correction with includes the cosmological constant (dark energy).

Chapter 2

Theoretical Cosmology

Now that we have a reasonable understanding of the observational aspects of cosmology, we shall move into the theoretical. To begin, let us gain a stronger understanding of cosmological redshift.

2.1 The Scale Factor of the Universe

Throughout the universe the distance between two points changes with time (as the universe expands) by a scale factor $a(t)$, where $a(t_0) = 1$. This means that we can rewrite the distances between two points as:

$$r_{12}(t) = a(t)r_{12}(0); r_{23}(t) = a(t)r_{23}(0); r_{13}(t) = a(t)r_{13}(0). \tag{2.1}$$

Using more general index notation:

$$r_{ij} = a(t)r_{ij}(0). \tag{2.2}$$

Taking the derivative gives the following:

$$v_{ij} = \frac{d}{dt}r_{ij}(t) = \dot{a}(t)r_{ij} = \frac{\dot{a}}{a}r_{12}(0) \tag{2.3}$$

and then by Hubble's Law:

$$\frac{\dot{a}}{a} = H(t) \tag{2.4}$$

Let us additionally consider the change of wavelength over these distances:

$$\frac{\lambda_o}{a(t_0)} = \frac{\lambda_e}{a(t_e)} \tag{2.5}$$

where t_e is the time of emission and t_0 is the current time.

$$\frac{\lambda_0}{\lambda_e} = \frac{1}{a(t)} \tag{2.6}$$

$$\implies a(t) = \frac{1}{1+z}. \tag{2.7}$$

We have seen this equation before, and now we have a better understanding of where it comes from. More specifically, it is derived as a result of the expansion of space itself. Looking at the following cases:

- If $z = 1$:
$$a(t) = \frac{1}{2} \tag{2.8}$$
(half the current size of the Universe)
- If $z = 99$:
$$a(t) = \frac{1}{100} \tag{2.9}$$
(one-hundredth the current size of the Universe)

From this, it is clear that a larger z implies going back in time to when the universe was smaller.

2.2 Curvature

Clearly, the metric of space itself is important here, so let us quickly discuss curvature.[1] One denotes a flat universe as $k = 0$, a positively curved (finite) universe as $k = 1$, and a negatively curved (infinite) universe as $k = -1$. This can be summarised in the following table.

k value	curvature	ρ_{total}	shape
+1	positive	$> \rho_{crit}$	closed
0	flat	$= \rho_{crit}$	flat
-1	negative	$< \rho_{crit}$	open

2.2.1 2D Curvature

Flat Universe (k = 0):

$$\sum_{i}^{3} \theta_i = 180 \implies ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 \tag{2.10}$$

Positively Curved Universe (k = 1):

$$\sum_{i}^{3} \theta_i = 180 + A/R^2 \implies ds^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2 \tag{2.11}$$

Negatively Curved Universe (k = -1):

$$\sum_{i}^{3} \theta_i = 180 - A/R^2 \implies ds^2 = dr^2 + R^2 \sinh^2(r/R) d\theta^2 \tag{2.12}$$

2.2.2 3D Curvature

Flat Universe (k = 0):

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.13}$$

Positively Curved Universe (k = 1):

$$ds^2 = dr^2 + R^2 \sin^2(r/R) \left[d\theta^2 + \sin^2\theta d\phi^2\right] \tag{2.14}$$

Negatively Curved Universe (k = -1):

$$ds^2 = dr^2 + R^2 \sinh^2(r/R) \left[d\theta^2 + \sin^2\theta d\phi^2\right] \tag{2.15}$$

[1] A more rigorous introduction to this subject would require differential geometry and general relativity. I hope to expand into these areas in the future.

This can be rewritten as:

$$ds^2 = dr^2 + S_K^2(r)d\Omega^2 \tag{2.16}$$

where,

$$S_K(r) = \begin{cases} r & k = 0 \\ R\sin(r/R) & k = 1 \\ R\sinh(r/R) & k = -1 \end{cases} \tag{2.17}$$

2.3 Robertson Walker Metric

The Robertson Walker Metric includes both space and time:

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega \tag{2.18}$$

For photons, $ds^2 \to 0, d\Omega \to 0$, therefore

$$cdt = dt \tag{2.19}$$

or

$$\frac{dr}{dt} = c. \tag{2.20}$$

This is called a *"null geodesic."*

2.4 Friedmann Equation

[Insert derivation from Einstein Field Equations here]
After some derivation, we arrive at the Friedmann equation:

$$\begin{aligned} H^2(t) &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{(R_0a(t))^2} = H_0^2\left[\frac{\Omega_M(0)}{a^3} + \Omega_\Lambda(0) + \frac{\Omega_K(0)}{a^2} + \frac{\Omega_r(0)}{a^4}\right] \\ H^2(t) &= H_0^2\left[\frac{\Omega_r(0)}{a^4} + \frac{\Omega_M(0)}{a^3} + \Omega_\Lambda(0) + \frac{1-\Omega_0}{a^2}\right] \\ H_0 &= \frac{8\pi G}{3}\rho(0) - \frac{kc^2}{R_0} \end{aligned}$$

For completeness, I have listed some of the different forms of the equation. For now, let us consider the following variant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_E(t) - \frac{kc^2}{R_0^2a(t)^2}. \tag{2.21}$$

This equation tells us that if $\dot{a} > 0$, the universe is expanding, if $\dot{a} < 0$, the universe is collapsing, and if $\dot{a} = 0$, the universe is static.

2.5 Solutions of the Friedmann Equations

2.5.1 Single Component Universe

1. **For** $k = 0, \Omega_M = 1$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_m(0)a^{-1} \rightarrow \dot{a}^2 = \frac{H_0^2}{a} \rightarrow \dot{a} = H_0 a^{-1/2} \tag{2.22}$$

$$\implies a(t) = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \tag{2.23}$$

$$\int_0^1 a(t)dt = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \rightarrow t_0 = \frac{2}{3}H_0 \tag{2.24}$$

   This is too young because we have observed clusters that are older than this.

2. **For** $k = 0, \Omega_r = 1$

$$\dot{a}^2 = \frac{H_0}{a^2} \rightarrow \dot{a} = \frac{H_0}{t} \tag{2.25}$$

$$a^2 = 2H_0 t \tag{2.26}$$

$$a(t) = \sqrt{2H_0}t^{1/2}t^{1/2}; \quad t_0 = \frac{1}{2H_0} \tag{2.27}$$

3. **For** $k = 0, \Omega_\Lambda = 1$

$$\dot{a}^2 = H_0^2 a^2 \rightarrow \dot{a} = H_0 a \tag{2.28}$$

$$a(t) \propto e^{H_0 t} \tag{2.29}$$

2.5.2 Multi-Component Universe

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = H_0 \left[\frac{\Omega_M(0)}{a^3} + \Omega_\Lambda(0) + \frac{\Omega_K(0)}{a^2} + \frac{\Omega_r(0)}{a^4}\right]^{1/2} \tag{2.30}$$

This version of the Friedmann equation is the best way for analyzing multi-component universes. Let's look at some important facts first:

$$\Omega_M(t) = \frac{\rho_M(t)}{\rho_{crit(t)}} = \frac{\rho_M(0)a^{-3}}{\rho_{crit}(0)E(z)^2} \tag{2.31}$$

$$= \frac{\Omega_M(0)a^{-3}}{\left[\frac{\Omega_M(0)}{a^3} + \Omega_\Lambda(0) + \frac{\Omega_K(0)}{a^2} + \frac{\Omega_r(0)}{a^4}\right]} \tag{2.32}$$

Now noting that Ω_r and Ω_k are negligible, one gets:

$$\Omega_M(t) = \frac{1}{1 + \frac{\Omega_\Lambda a^3}{\Omega_M(0)}} \tag{2.33}$$

$$\implies \Omega_M(a \approx 0) \to 1. \tag{2.34}$$

1. For a **Flat Universe**, $k = 0$, we know that the universe expands at a rate $a(t) \propto t^{2/3}$
2. For a **Curved Universe** $k = \pm 1$, with $\Omega_\Lambda = \Omega_r = 0$:

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_M(0)}{a^3} + \frac{(1 - \Omega_M(0))}{a^2} \tag{2.35}$$

$\Omega_M(0) > 1$ (closed universe, $k = 1$) $\implies$ recollapse.
$\Omega_M(0) < 1$ (open universe, $k = -1$) $\implies$ expansion.

$$a_{max} = \frac{\Omega_M(0)}{\Omega_M(0) - 1} \tag{2.36}$$

Therefore, the more mass we have the quicker we recollapse.

3. **Matter + Λ only**

$$\implies \Omega_M + \Omega_\Lambda = 1 \tag{2.37}$$

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_M(0)}{a^3} + \Omega_\Lambda(0) \tag{2.38}$$

2.6 ΛCDM Model of the Universe

This is the widely accepted model which we think explains our universe as it exists today. It's Ω components are: $\Omega_M(0) = 0.3, \Omega_\Lambda(0) = 0.7$ and $\Omega_K(0) = 0$. We also use $H_0 = 70 \pm 3$ kms^{-1}Mpc^{-3}, which has been measured by a variety of different methods.

We know that $\Omega_M(0) \approx 0.25 - 0.30$ because:

- The luminosity function tells us that $\rho_M(0) \approx 0.3$
- We have measured the number of baryons in clusters: $\frac{\Omega_{baryons(0)}}{\Omega_{clusters}} = 0.15$
- $\frac{\Omega_{baryons}(0)}{\Omega_M(0)} = \frac{0.045}{\Omega_M(0)}$
- The mass function of clusters is closely related to the mass density of the universe $\implies \frac{0.045}{\Omega_M(0)} = 0.15$ $\Omega_M(0) = 0.30$

We know that $\Omega_\Lambda = 0.70$ because:

- it has been found from type 1a supernova.
- we have observed it in the CMB.

2.7 Epochs of the Universe

Event	Redshift, z	Temp, T (K)	kT (Energy)	Age
Today	0	2.7	0.23 MeV	13.7 Gyr
M-Λ	≈ 0.4	≈ 4		10 Gyr
Reionization	≈ 10	30		
First Star formation	$\approx$ 10 - 20			$<$ 1 Gyr
Recombination	≈ 1100	3000		390 Kyr
M-Rad	≈ 3500	10,000	0.83 eV	50 Kyr
B.B.N	$\approx 3 \times 10^5$	10^{10}	1 MeV	3 minutes
Quark Formation	10^{12}	3×10^{12}	300 MeV	
GUT		10^{28}	10^{12} TeV	
Planck scale		10^{32}	10^{16} TeV	10^{-44} seconds

2.8 Distances

2.8.1 Proper distance and horizon distance

- Proper distance at the current time, t_0:

$$d_p(t_0) = \int_{t_e}^{t_0} \frac{cdt}{a(t)} = \int_0^z \frac{cdz}{H(t)} = \int_0^z \frac{cdz}{H_0 E(z)} \tag{2.39}$$

  this is the comoving distance.

- Proper distance at time of emission t_e:

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} \tag{2.40}$$

- Horizon distance:

$$d_H = \int_0^\infty \frac{cdz}{H_0 E(z)} \tag{2.41}$$

  The Hubble radius, $R_H = \frac{c}{H_0} \approx 4500\text{Mpc}$

$\Omega_M(0) = 1$ **(flat)**

$$d_p(t_0) = \int_0^z \frac{cdz}{H_0 E(z)} = \int_0^z \frac{c}{H_0(1+z)^{3/2}} dz \tag{2.42}$$

$$= \frac{-2c}{H_0} \left[(1+z)^{-1/2}\right]_z^0 \tag{2.43}$$

$$= \frac{2c}{H_0} \left[1 - \frac{1}{\sqrt{1+z}}\right] \tag{2.44}$$

therefore, as $z \to \infty$, $d_p = d_H \to \dfrac{2c}{H_0}$

$\Omega_\Lambda(0) = 1$ **(Flat)**

$$d_p(t_0) = \int_0^z \frac{cdz}{H_0} = \frac{c}{H_0} z \tag{2.45}$$

which means that as $z \to \infty, d_p = d_H \to \infty$.

ΛCDM **Model**
We know from measurement that $d_H = \dfrac{3.24c}{H_0} \approx 14{,}000$ Mpc.
Therefore:

$$d_H = \frac{c}{H_0} \frac{2}{1+3\omega} \tag{2.46}$$

provided that $\omega > -1/3$.

2.8.2 Luminosity Distance

$$d_L = d_L(t_0)(1+z) \tag{2.47}$$

2.8.3 Angular Distance

$$d_\theta = \frac{R}{\theta} = d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{d_L}{(1+z)^{1/2}} \tag{2.48}$$

as $z \to \infty$, $d_L, d_\theta, d_p(t_0) \to \frac{cz}{H_0}$
as $z \to 0$

- $d_p(t_0) \to d_H(finitevalues)$
- $d_L \to z d_H \to \infty$
- $d_\theta \to \frac{d_H}{z} \to 0$

2.9 Thermal History

Let us consider the blackbody radiation density, ρ_{rad}

$$\rho_{rad} \propto T^4 \propto a^{-4} \propto (1+z)^4 \tag{2.49}$$

$$T(z) = T_0(1+z)T(a) = T_0/a \tag{2.50}$$

where $T_0 = 2.725$K.
So for example, $z = 3000, T(z) = 8000$K

2.10 Cosmic Microwave Background

The universe was opaque up until $z = 1100$ when recombination and photon decoupling occurred. From $z = 1100 \to z = 0$, the universe became transparent. This implies that when we look at the CMB, we are seeing the radiation that came from the moment the universe became transparent ("the surface of last scattering"). The differences in color of figure (2.7) are minute temperature differences. The current temperature of the CMB is $T(z=0) = 2.7$K and therefore $T(z = 1100) = 2.7(1+1100) \approx 3000$K (with small deviations in the temperature due to overdensities). The surface of last scattering is a perfect black body.

2.10.1 Baryonic Acoustic Oscillations

These are regular, periodic fluctuations in the density of visible baryonic matter. [to be added to]

2.11 Big Bang Nucleosynthesis (BBN)

BBN concentrates on the first 3 minutes of the Universe. The content of the volume of the early universe contains photons, neutrinos, Dark Matter, Dark Energy, protons and electrons. The number density of these decrease over time, but photons dominate in the early universe. Just 1 second after the Big Bang, the temperature was 10^{10}K, corresponding to an energy of 1 MeV, but 10^{-4} seconds after the big bang, the temperature was 10^{12}K with an energy of 100MeV.

Given that we have both electron, e^- and positrons, e^+ at this time, the following annihilation is taking place (at t = 10^{-4}):

$$e^+ + e^- \rightleftharpoons \gamma + \gamma \tag{2.51}$$

But at $t = 1$s and later, the equation is only forward:

$$e^+ + e^- \to \gamma \tag{2.52}$$

because photon energy is too weak to do the reverse.

The following is also taking place:

$$p + \bar{p} \rightleftharpoons \gamma + \gamma \tag{2.53}$$

So there must have been an imbalance such that we are left in a matter dominated universe, with little antimatter.

2.11.1 Beta Decay

The equation for β decay is as follows:

$$n \to p + e^- + \bar{\nu_e} \tag{2.54}$$

Note that beta decay is mediated by the weak force. The half life of a neutron is 900 seconds. The reverse reaction can also happen:

$$p + e^- \to n + \nu_e \tag{2.55}$$

As the universe cools, protons and neutrons start moving slower, and eventually form the light elements of the periodic table. Currently, Helium makes up $\approx 25\%$ of the visible matter in the universe, and Helium makes up $\approx 75\%$ of the visible matter. Heavier elements (such as iron) are produced by fusion.

2.12 Inflation

There are four main 'problems' in the universe:

1. Flatness problem: Why is K = 0?
2. Horizon problem: why is the universe homogeneous and isotropic on large scales?
3. Monopole problem: why don't we see magnetic monopoles when we see electric monopoles?
4. What are the causes of the anisotropies in the CMB?

Let us begin by further discussing the flatness problem.

2.12.1 The Flatness Problem

Starting with the Friedmann equation in density parameter form:

$$1 = \Omega(t) = \frac{-kc^2}{R_0 a(t)^2 H(t)^2}. \tag{2.56}$$

At the present time:

$$1 - \Omega_0 = \frac{-kc^2}{R_0^2 H_0^2} \tag{2.57}$$

Combining equation (2.56) and (2.57) gives us

$$1 - \Omega(t) = \frac{H_0^2(1-\Omega_0)}{H(t)^2 a(t)^2}. \tag{2.58}$$

We also know that during the period when the universe was dominated by matter and radiation:

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_r(0)}{a^4} + \frac{\Omega_M(0)}{a^3}. \tag{2.59}$$

Thus the density parameter evolved as

$$1 - \Omega(t) = \frac{(1-\Omega_0)a^2}{\Omega_r(0) + a\Omega_M(0)}. \tag{2.60}$$

During the period when the universe was dominated by radiation, the deviation of Ω from 1 was constantly growing. During the radiation dominated phase:

$$|1 - \Omega|_r \propto a^2 \propto t \tag{2.61}$$

and during the matter-dominated phase:

$$|1 - \Omega| \propto a \propto t^{2/3}. \tag{2.62}$$

Given these relations we know the accuracy of $|1 - \Omega|$ at various times.

At matter-radiation equality:

$$|1 - \Omega_{rM}| \leq 2 \times 10^{-4}. \tag{2.63}$$

At BBN:

$$|1 - \Omega_{BBN}| \leq 3 \times 10^{-14}. \tag{2.64}$$

And finally at plank time:

$$|1 - \Omega_P| \leq 1 \times 10^{-60}. \tag{2.65}$$

This is extremely flat to a high degree of accuracy. This raises the question: what could be the cause of this flatness?

2.12.2 The Horizon Problem

Despite the CMB being so consistent in temperature all across (2.726 ± 0.001 K), various points of the CMB are not causally connected. The horizon distance at the time of the surface of last scattering was 0.4 Mpc. Since the angular diameter distance to the last scattering surface is 13 Mpc, the angular separation is:

$$\theta_{hor} = 2^o \tag{2.66}$$

Therefore points on the last scattering surface were separated by an angle of 2^o. They could not be on contact. This raises the question then: why is the universe homogeneous and isotropic?

2.12.3 Inflation as a solution?

In order to solve the above problems, a theory known as cosmic inflation was proposed, which sees rapid expansion of the universe very early on. Here is an introduction to the formalism:
At the time inflation began, the Hubble constant $H_i =\approx 10^{36}\text{s}^{-1}$, and $t_i = 10^{-36}$s.

$$a(t) = \begin{cases} a_i \left(\dfrac{t}{t_i}\right)^{1/2} & t < t_i \\ a_i e^{H_i(t-t_i)} & t_i < t < t_f \\ a_i e^{H_i(t-t_i)} \left(\dfrac{t}{t_f}\right)^{1/2} & t > t_f \end{cases} \tag{2.67}$$

Let us now define e-folding. e-folding is the time interval in which an exponentially growing quantity increases by a factor of e. Let's consider the *e-folding* time of inflation

$$\frac{a(t_f)}{a(t_i)} = e^{H_i(t_f - t_i)} \equiv e^{\mathcal{N}} \tag{2.68}$$

where $\mathcal{N}$ is the *e-folding* time.

$$\begin{aligned} \mathcal{N} &= \frac{t_f - t_i}{t(H_i)} \\ &= \frac{10^{-34}}{10^{-36}} \\ &= 100 \end{aligned}$$

Since $e^{100} \approx 10^{43}$, the universe grew by 10^{43} order of magnitude during inflation.
It is worth noting that inflation does have its critics, and discussing these criticisms shall be the next part of this guide.
[To be continued]

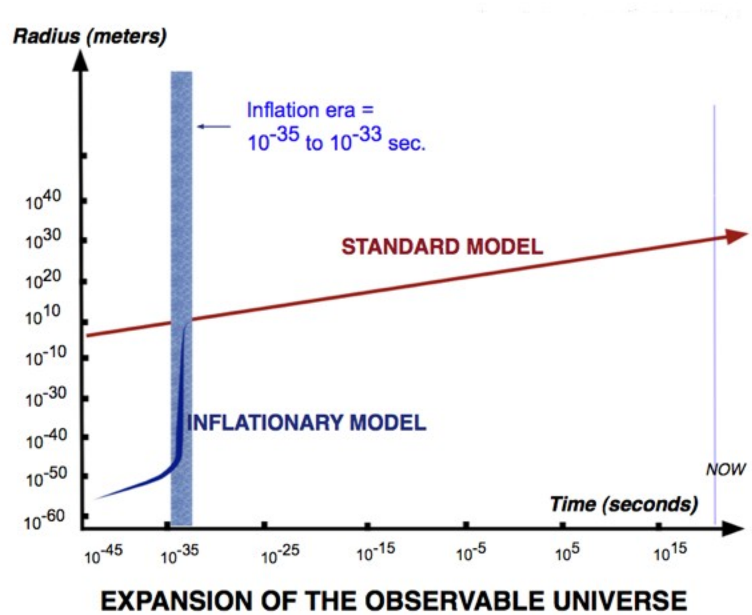

Figure 2.1: A graph showing expansion with inflation, in comparison to a standard model without inflation